The Signaling Dimension and the No-Hypersignaling Principle

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Motivation

General probabilistic theories (GPTs): description of systems in terms of states, effects, and a composition rule to get proabilities.

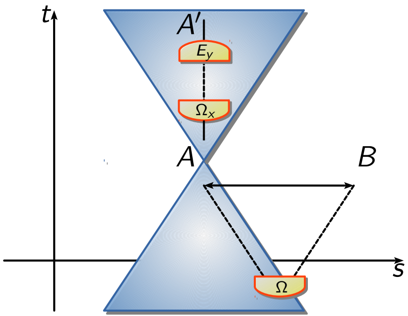
Quantum theory (QT), standard formulation: **mathematical** (Hilbert space) formalism without direct operational interpretation.

Motivation: to go in the direction of singling out QT based on principles that are:

- **operational**, rather than just mathematical;
- optionally, **device-independent** (DI).

DI principle: for **any** theory that violates it, the violation can be detected by only observing correlations among space-time events.

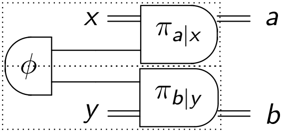
Space- vs Time-like correlations



- A, B: **space-like** separated, share space-like correlations Ω, no-signaling applies.
- A, A': **time-like** separated, information encoded into $\{\Omega_x\}$ and decoded by $\{E_y\}$, no-signaling does not apply.

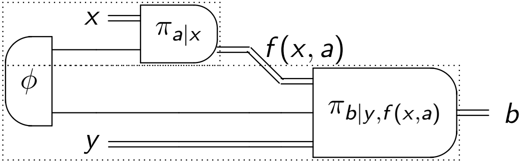
Principles constraining space-like correlations

Principles constraining **space-like** correlations are **DI**, since no-signaling allows any violation to be detected by a DI test:

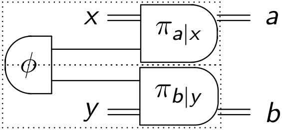


Principles constraining space- and time-like correlations

Information causality (IC) constrains correlations $p(b|x, y)$ by exchanging a limited amount of classical information $f(x, a)$:



A **sufficient condition** for IC violation can be obtained in a DI way by performing a *purely space-like* test, i.e. a Bell test:



After $p(a, b|x, y)$ is collected, one checks if there exists a postprocessing $y, b, f(x, a) \to b'$ such that $p(b'|x, y)$ violates IC.

A principle constraining purely time-like correlations

Motivation: constraining space-like correlations does not single out QT: what about constraining purely **time-like** correlations?

Framework characterizing time-like correlations of any given GPT:

- Davies-like theorem: violations attained by ray-extremal measurements.
- Characterization of extremal measurements.

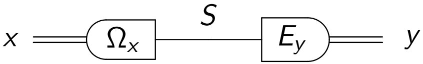
Applictions: full characterization of squit (square bit) bipartitions. Derivation of a squit bipartition that:

- is compatible with QT at the level of **space-like** correlations.
- displays an anomalous behavior in its **time-like** correlations.

Anomaly formalized by **no-hypersignaling** principle constraining **time-like** correlations, hence not DI.

Notation

Purely time-like setup (memory): upon input of x, Alice prepares system S into state Ω_x and transmits it to herself in the future:



Correlations set $\mathcal{P}_S^{m \to n}$: convex hull of all m-input/n-output conditional probability distributions $p_{y|x}$ achievable by system S.

$S = C_d, Q_d$ for d-dimensional classical or quantum system.

Frenkel and Weiner's theorem

Frenkel and Weiner recently proved this remarkable result[1]:

$$\mathcal{P}_{C_d}^{m \to n} = \mathcal{P}_{Q_d}^{m \to n}, \quad \forall m, n.$$

Holevo bound: C_d and Q_d achieve same mutual information.

Holevo bound constrains a **specific function** defined on $\mathcal{P}_{Q_d}^{m \to n}$.
Frenkel-Weiner's theorem constrains $\mathcal{P}_{Q_d}^{m \to n}$ itself.

The Holevo bound is a **corollary** of Frenkel-Weiner's theorem!

[1] P.E. Frenkel and M. Weiner, Commun. Math. Phys. **340**, 563 (2015)

Signaling dimension

Definition (Signaling dimension)

The **signaling dimension** *of a system S, denoted by $\kappa(S)$, is the smallest integer d such that $\mathcal{P}_S^{m\to n} \subseteq \mathcal{P}_{C_d}^{m\to n}$, $\forall m, n$.*

Properties of the signaling dimension:

- by definition, $\kappa(S)$ equals the usual classical dimension,
- $\kappa(S)$ also equals [2] the usual quantum dimension, thus for brevity $\mathcal{P}_d^{m\to n} := \mathcal{P}_{C_d}^{m\to n} = \mathcal{P}_{Q_d}^{m\to n}$,
- $\kappa(S)$ does not depend on an arbitrarily made choice of a specific protocol (such as perfect state discrimination);
- $\kappa(S)$ is non-trivial even for those theories where perfectly discriminable states do not exist.

[2] P.E. Frenkel and M. Weiner, Commun. Math. Phys. **340**, 563 (2015)

Applications of the signaling dimension

- Characterization [3] of locally classical GPTs.
- Characterization [4] of the polytope $\mathcal{P}_d^{m \to n}$.
- Characterization [5] of the signaling dimension for noisy channels.
- Relation [6] to the information storing capacity.

[3] G. M. D'Ariano, M. Erba, and P. Perinotti, Phys. Rev. A **101**, 2020; G. M. D'Ariano, M. Erba, and P. Perinotti, Phys. Rev. A **102**, 2020

[4] B. Doolittle and E. Chitambar, Phys. Rev. Research **3**, 2021; E. Chitambar, I. George, B. Doolittle, and M. Junge, IEEE Trans. Inf. Theory **69**, 1660 (2023)

[5] P. E. Frenkel and M. Weiner, Quantum **6**, 2022; P. E. Frenkel, Quantum **6**, 2022

[6] K. Matsumoto and G. Kimura, arXiv.1802.01162, 2018

Characterization of extremal measurements

Any measurement can be regarded as a probability distribution over normalized effects, that is, effects whose projection on the unit effect $\overline{e}$ equals $\overline{e}$.

Lemma (Extremal measurements with ray-extremal effects)

For any measurement $M = \{p_y > 0, e_y\}$ *with extremal normalized effects* $\{e_y\}$*, the following conditions are equivalent:*

1. M *is extremal,*
2. $\{e_y\}$ *are linearly independent.*

Computation of extremal measurements

A probability distribution p is a measurement on the extremal normalized effects given by the columns of matrix E if and only if

$$Ep = \overline{e},$$
$$p \geq 0.$$

That is, p is an extremal measurement if and only if it is a vertex of such a polytope, given through its faces description.

Proposition

The extremal measurements with extremal normalized effects can therefore be found [7] *by passing from the faces description to the vertices description, a standard problem that can be solved e.g. with the double description method.*

[7] M. Dall'Arno, A. Tosini, F. Buscemi, *The signaling dimension in generalized probabilistic theories*, arXiv:2311.13103

The size of the problem

Lemma

The number V of vertices of $\mathcal{P}_d^{m \to n}$ is given by

$$V = \sum_{k=1}^{d} k! \binom{n}{k} \left\{ {m \atop k} \right\}.$$

In typical instances of the problem, V is too large for te problem to be practically tractable!

Computation of the signaling dimension

- Any row of p that is the convex combination of other rows can be eliminated without altering the result, thus reducing the effective value of m (and thus V) without loss of generality.
- Any vertex of $\mathcal{P}_d^{m \to n}$ that contains an entry equal to one where p contains a zero will not contribute to the convex decomposition of p; hence can be discarded without loss of generality.

No-hypersignaling principle

Definition (No-hypersignaling principle)

A theory is **non-hypersignaling** *iff, for any set of systems* $\{S_k\}$, *the signaling dimension of* $\otimes_k S_k$ *satisfies*

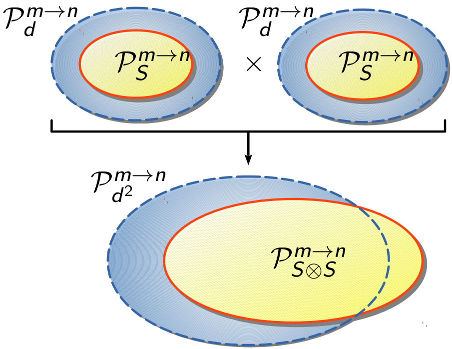
$$\kappa(\otimes_k S_k) \leq \prod_k \kappa(S_k).$$

Informally, in terms of input-output correlations, it must not matter if the systems are transmitted separately or jointly.

Even more informally, the no-hypersignaling is a purely time-like counterpart of no-signaling.

For two identical systems $\mathcal{P}_S^{m\to n} \subseteq \mathcal{P}_d^{m\to n} \implies \mathcal{P}_{S^{\otimes 2}}^{m\to n} \subseteq \mathcal{P}_{d^2}^{m\to n}$.

A hypersignaling theory

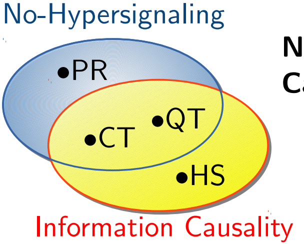


An example of a **hypersignaling theory**: while system S satisfies $\mathcal{P}_S^{m \to n} \subseteq \mathcal{P}_d^{m \to n}$, and thus has signaling dimension d, the composite system $S \otimes S$ has a signaling dimension strictly larger than d^2.

Informally, by transmitting the systems jointly rather than separately, better correlations are achieved.

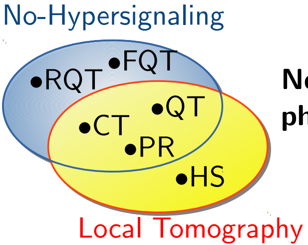
Conclusion

Is no-hypersignaling independent of other principles?



No-Hypersignaling vs Information Causality:

- **CT** classical theory,
- **QT** quantum theory,
- **PR** PR-boxes theory,
- **HS** hypersignaling theory.

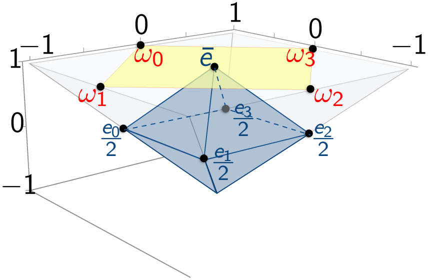


No-Hypersignaling vs Local Tomography:

- **RQT** real quantum theory,
- **FQT** fermionic quantum theory.

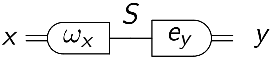
A toy model theory...

We start from the **squit**, the elementary system of the theory commonly considered to produce PR correlations:



- four extremal **states** $\{\omega_x\}$ (yellow square),
- four extremal **effects** $\{e_y\}$, plus the null and unit effects $0, \bar{e}$ (blue cone).

For a squit S one has that $\mathcal{P}_S^{m\to n} = \mathcal{P}_2^{m\to n}$, that is any correlation $p_{y|x}$ achievable by transferring a squit:



is also achievable by transferring a classical bit.

All of the bipartitions of a squit

All bipartite extensions of a squit can be given in terms of:

- 24 extremal bipartite states, 8 of which are entangled,
- 24 extremal bipartite effects, 8 of which are entangled.

We derived all the self-consistent bipartite extensions of a squit:

1. **PR model:** All the 24 states; only the 16 factorized effects;
2. **HS model:** Only the 16 factorized states; all the 24 effects;
3. **Frozen Models:** Only one entangled state and effect included, but no allowed reversible dynamics.

Since, for PR model, extremal measurements have four effects, $\mathcal{P}_{S \otimes S}^{m \to n} = \mathcal{P}_{4}^{m \to n}$, i.e. PR model cannot violate no-hypersignaling.

Analogously, since HS model has no entangled states, it cannot exhibit superclassical space-like correlations.

Extremal measurements of two squits

M	#	E_0	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}	E_{11}	E_{12}	E_{13}	E_{14}	E_{15}	E_{16}	E_{17}	E_{18}	E_{19}	E_{20}	E_{21}	E_{22}	E_{23}
0	2																	120		120					
1	4	60		60						60		60													
2	4	60		60							60		60												
3	6	30	30									30	30							60					60
4	6	30					30					30					30					60			60
5	6	40										40							40	40		40			40
6	7	30	30					30		30		30					30								60
7	8	20	20			20						40					20			40		40			40
8	8	20	20					40		20	20						40	40							40
9	8	40	20					20			20		40			20				40					40
10	8	30					30						30			30				30	30	30			30
11	9	20	20			20		20			20	20					40					40			40
12	9	15	15			15		30			30						45	30				30			30
13	9	20	20			20			20			20	20		20	20				80					
14	9	24		24			24						48		24					24	24			24	24
M	#	E_0	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}	E_{11}	E_{12}	E_{13}	E_{14}	E_{15}	E_{16}	E_{17}	E_{18}	E_{19}	E_{20}	E_{21}	E_{22}	E_{23}

Does the HS model violate the no-hypersignaling principle?

Consider payoff g and input/output correation p, achievable by transmitting a family of seven states $\{\Omega_x\}$ and performing a measurement with seven effects $\{E_y\}$ of HS model:

$$g = \frac{1}{21}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 \end{pmatrix}, \quad p = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix},$$

By exchanging two classical bits, the optimal payoff is $10/21$, but $g \cdot p = 1/2 > 10/21$, thus no-hypersignaling is **violated**!

The signaling dimension of two squits is five

M	#	**d**	**g · b**	**v**	**V**
3	6	4		128	$\sim 9 \cdot 10^{10}$
4	6	4		64	$\sim 9 \cdot 10^{10}$
5	6	4		465	$\sim 9 \cdot 10^{10}$
6	7	5	2	672	$\sim 4 \cdot 10^{12}$
7	8	5	$1/3$	60752	$\sim 10^{13}$
8	8	5	$8/3$	7616	$\sim 10^{13}$
9	8	5	2	10040	$\sim 10^{13}$
10	8	4		576	$\sim 3 \cdot 10^{11}$
11	9	5	$4/3$	37136	$\sim 2 \cdot 10^{13}$
12	9	5	2	107504	$\sim 2 \cdot 10^{13}$
13	9	5	$2/3$	8704	$\sim 2 \cdot 10^{13}$
14	9	5	$8/5$	488092	$\sim 2 \cdot 10^{13}$
M	#	**d**	**g · b**	**v**	**V**

Corollary

The signaling dimension of the composition of two squits named HS model, including all eight possible entangled effects, is five.

Motivation: characterizing time-like correlations allowed by QT.

Our general **program**[8]:

- We introduced signaling dimension as an *operational*, *task-independent* dimension for *any* GPT.
- We derived a general *theoretical framework* for the computation of the signaling dimension.
- We introduced the no-hypersignaling principle as a *scaling rule* for signaling dimension under system composition.

Applications:

- We *fully* characterized *all* the bipartite extensions of a squit.
- By applying our framework, we showed that the HS model's time-like correlations violate the no-hypersignaling principle, but its space-like correlations are compatible with CT and QT.

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[8] M. Dall'Arno, S. Brandsen, A. Tosini, F. Buscemi, and V. Vedral, *No-hypersignaling principle*, Phys. Rev. Lett. **119**, 020401 (2017)