When is Maxwell's demon consistent with the second law?

Universal validity of the second law of information thermodynamics

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arXiv: 2308.15558

1. Introduction

The 2^{nd} law of thermodynamics

- $\Delta S \geq 0$ (isolated or adiabatic process)

- $W_{\mathrm{ext}} \leq -\Delta F_{\mathrm{eq}}$ (isothermal process)

1. Introduction

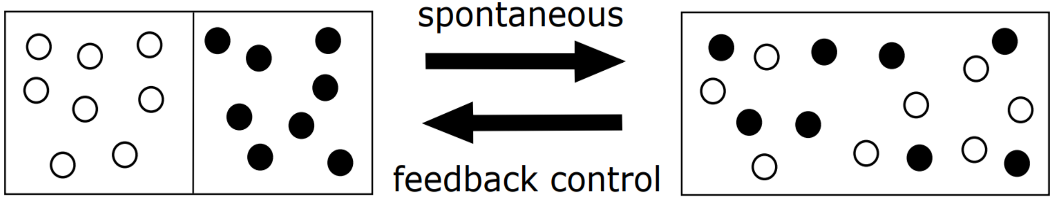

Fig.1 Maxwell's demon[1]

[1] J. C. Maxwell, Theory of heat (Appleton, London, 1871)

The 2nd law of information thermodynamics (Sagawa and Ueda [3])

[2] $W_{\mathrm{fb}} \leq -\Delta F_{\mathrm{eq}} + \beta^{-1} I_{\mathrm{QC}}$ ※$I_{\mathrm{QC}} := S(\rho_0^A) - \sum_k p_k S\left(\frac{\sqrt{\mathrm{A}_k}\rho\sqrt{\mathrm{A}_k}}{\mathrm{Tr}[\mathrm{A}_k \rho]}\right)$

[3] $W_{\mathrm{meas}} + W_{\mathrm{eras}} \geq \beta^{-1} I_{\mathrm{QC}}$

[2] T. Sagawa and M. Ueda, Phys. Rev. Lett. 100, 080403 (2008)
[3] T. Sagawa and M. Ueda, Phys. Rev. Lett. 102, 250602 (2009)
[4] T. Sagawa and M. Ueda, Phys. Rev. Lett. 106, 189901 (2011) (the erratum of [3])

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いらすとや

Sagawa and Ueda[2-4] imposed some assumptions to the measurement and feedback control

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Question

When is Maxwell's demon consistent with the 2nd law?

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[2] T. Sagawa and M. Ueda, Phys. Rev. Lett. 100, 080403 (2008)
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<u>Question</u>
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<u>Approach</u> S. M., M. H. Mohammady, K. Sakai, K. Kato, and F. Buscemi, arXiv: 2308.15558 (2023)
Generalize measurement and feedback control while keeping the minimal structure of Maxwell's demon

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[2] $W_{\rm fb} \leq -\Delta F_{\rm eq} + \beta^{-1} I_{\rm QC}$ ※$I_{\rm QC} := S(\rho_0^A) - \sum_k p_k S\left(\frac{\sqrt{\mathrm{A}_k}\rho\sqrt{\mathrm{A}_k}}{\mathrm{Tr}[\mathrm{A}_k\rho]}\right)$

[3] $W_{\rm meas} + W_{\rm eras} \geq \beta^{-1} I_{\rm QC}$

Approach

Generalize measurement and feedback control while keeping the minimal structure of Maxwell's demon

Answer S. M., M. H. Mohammady, K. Sakai, K. Kato, and F. Buscemi, arXiv: 2308.15558 (2023)

The 2nd law of information thermodynamics is *universal* in a sense that…

2. Basic notions

◆ Some quantities (all of them are non-negative, e.g., [7])

- Von Neumann entropy [5]: $S(\rho^A)$ or $S(A)_\rho := -\mathrm{Tr}[\rho^A \ln \rho^A]$
- Umegaki relative entropy [6]: $D(\rho^A||\sigma^A) \coloneqq \mathrm{Tr}[\rho^A \ln \rho^A - \rho^A \ln \sigma^A]$ $(\sigma^A > 0)$
- Quantum mutual information: $I(A:A')_\rho \coloneqq S(A)_\rho + S(A')_\rho - S(AA')_\rho$
- Conditional quantum mutual information:

$$I(A:B|C)_\rho \coloneqq S(A|C)_\rho + S(B|C)_\rho - S(AB|C)_\rho$$

[5] J. von Neumann, Mathematical foundations of quantum mechanics (Princeton University Press, 1955).
[6] H. Umegaki, Proc. Japan Acad. 37, 459 (1961).
[7] M. M. Wilde, Quantum Information Theory, 2nd edition (Cambridge University Press, 2017).

2. Basic notions

◆Thermodynamic systems Y

- Hamiltonian: H^Y
- Averaged energy: $E(\rho^Y; H^Y) := \mathrm{Tr}[\rho^Y H^Y]$
- Inverse temperature: $\beta := \frac{1}{kT} > 0$
- Partition function: $Z^Y := \mathrm{Tr}\, e^{-\beta H^Y}$
- Thermal state (equilibrium): $\gamma^Y := \frac{e^{-\beta H^Y}}{Z^Y}$

2. Basic notions

◆Free energy

- Equilibrium free energy: $F_{\mathrm{eq}}^{Y} \coloneqq -\beta^{-1} \ln Z^{Y}$
- Nonequilibrium free energy [8]:

$$F(\rho^{Y}; H^{Y}) \coloneqq E(\rho^{Y}; H^{Y}) - \beta^{-1} S(Y)_{\rho} = F_{\mathrm{eq}}^{Y} + \beta^{-1} D(\rho^{Y} || \gamma^{Y})$$

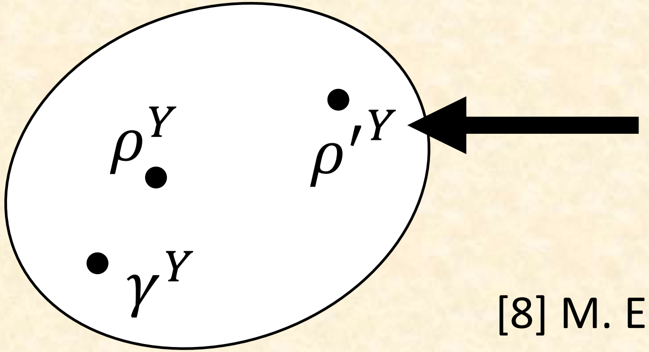

Fig. 2. ρ'^{Y} is further from γ^{Y}, more nonequilibrium, $F(\rho^{Y}; H^{Y}) \leq F(\rho'^{Y}; H^{Y})$

[8] M. Esposito and C. Van den Broeck, EPL 95, 40004 (2011).

2. Basic notions

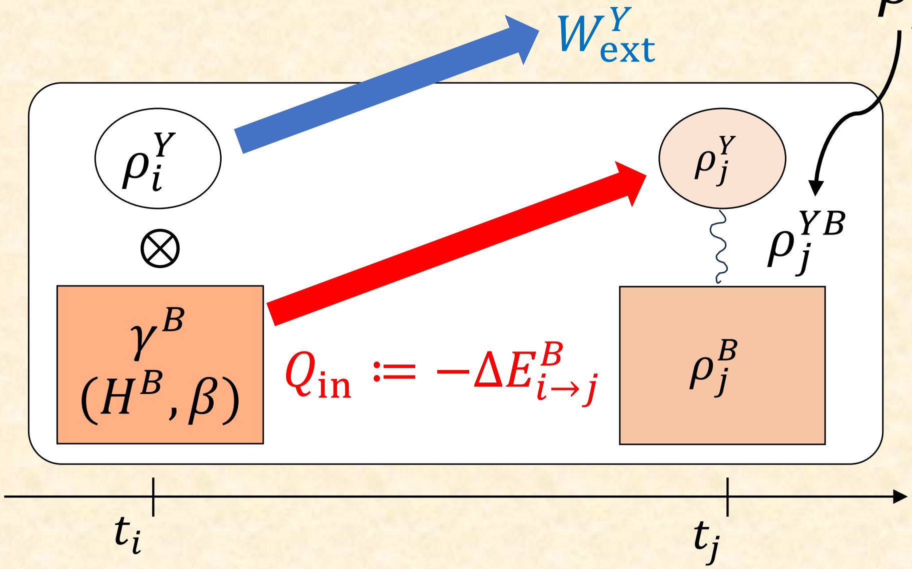

Fig. 3. Isothermal quantum information processing

$$\rho_j^{YB} = U(\rho_i^Y \otimes \gamma^B)U^\dagger$$

Nonequilibrium 2nd law [8]

$$W_{\text{ext}}^{Y} \leq -\Delta F_{i\to j}^{Y}$$

[8] M. Esposito and C. Van den Broeck, EPL 95, 40004 (2011).

$$W_{\text{ext}}^{Y} = -\Delta E_{i\to j}^{Y} + Q_{\text{in}} \quad \text{(1st law)}$$

$$= -\Delta E_{i\to j}^{YB} = -\Delta F_{i\to j}^{Y} - \beta^{-1} S_{\text{irr}}$$

$$S_{\text{irr}} \coloneqq I(Y:B)_{\rho_j} + D(\rho_j^B \| \gamma^B) \geq 0$$

(*irreversible entropy production*)

See, S. M., M. H. Mohammady, K. Sakai, K. Kato, and F. Buscemi, arXiv: 2308.15558 (2023)

2. Basic notions

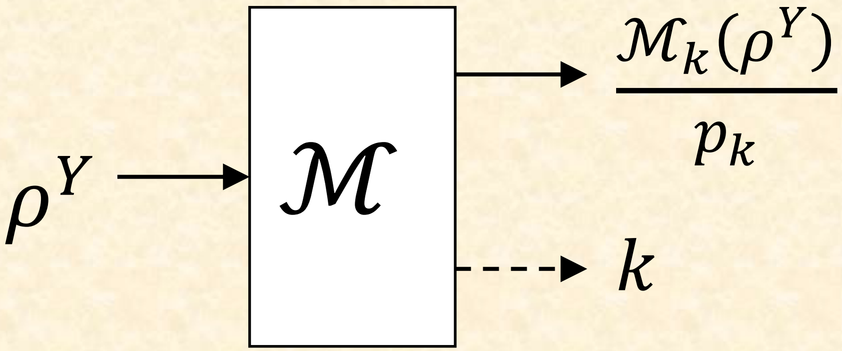

Fig. 4. CP-instrument [9]

$\mathcal{M}_k$: Completely Positive Trace Non-Increasing linear map (assume the same input and output)

$\sum_k \mathcal{M}_k$: Trace-Preserving

[9] M. Ozawa, J. Math. Phys. 25, 79 (1984).

Kraus rep. of $\mathcal{M}_k$

$$\mathcal{M}_k(\cdot) = \sum_i \Lambda_k^{(i)}(\cdot)\Lambda_k^{(i)\dagger}$$

CP-instrument $\mathcal{M}$ is *efficient* [10] if for all k, $\mathcal{M}_k(\cdot) = \Lambda_k(\cdot)\Lambda_k^{\dagger}$ ※

※ The word *efficient* appears in [11] is in a different meaning (*quasicomplete* [12]).

[10] e.g., K. Jacobs, Phys. Rev. A 80, 012322 (2009).
[11] H. M. Wiseman, Quantum Trajectories and Feedback, Ph.D. thesis (1994).
[12] M. Ozawa, J. Math. Phys. 27, 759 (1986).

2. Basic notions

- Indirect measurement model [9]

[9] M. Ozawa, J. Math. Phys. 25, 79 (1984).

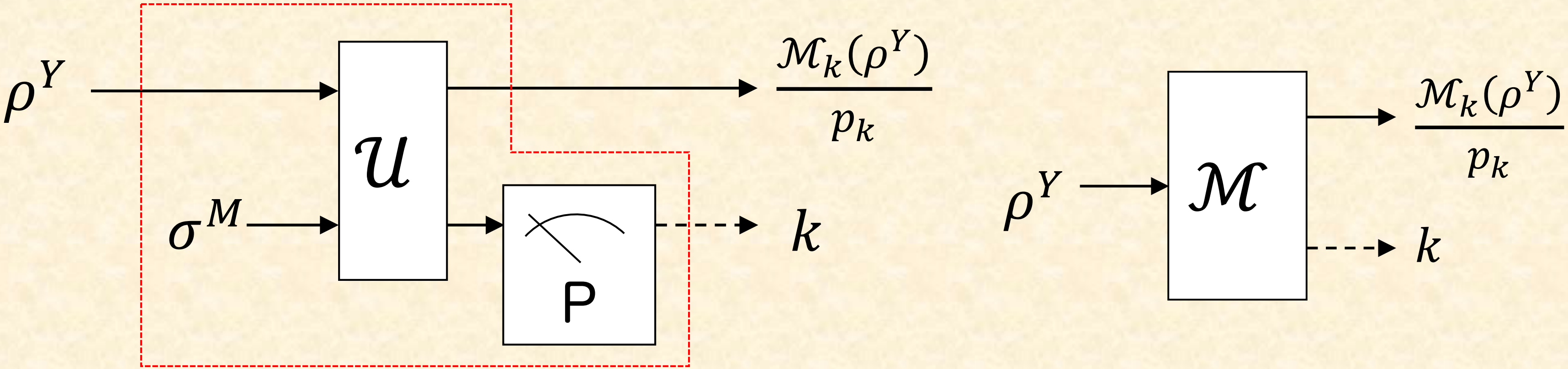

Fig. 5. Left: Indirect measurement characterized by $(\mathcal{H}^M, \sigma^M, \mathcal{U}, \mathrm{P})$. Right: CP-instrument

3. Feedback control and erasure

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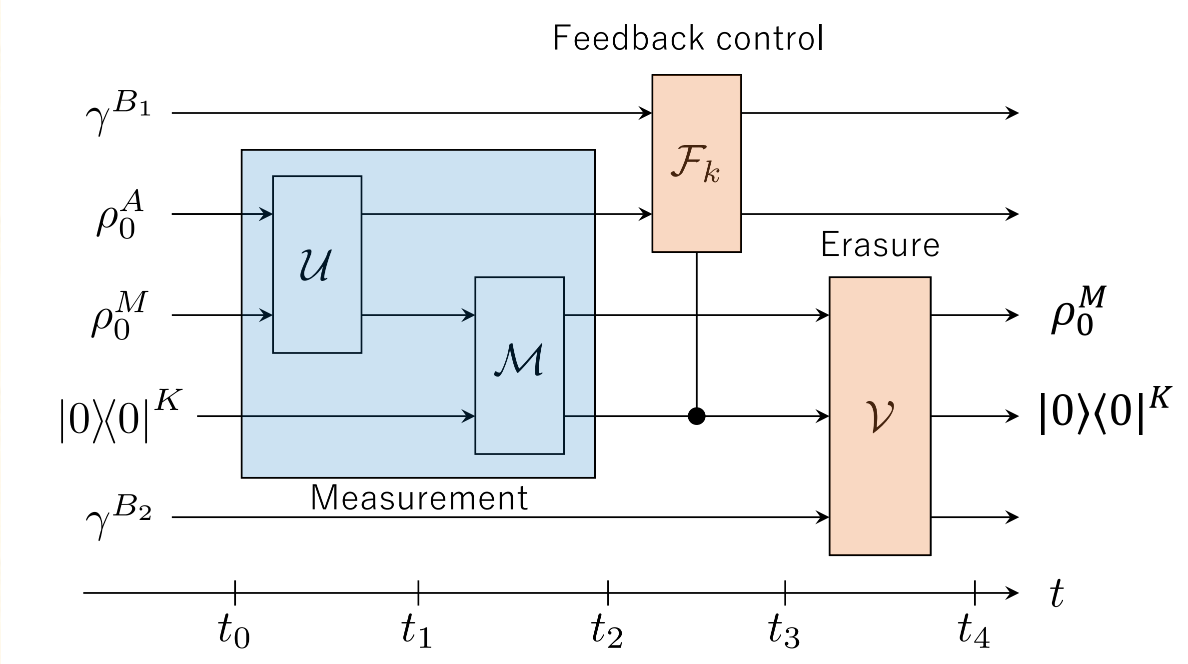

Fig. 6. Feedback control and erasure protocol

3. Feedback control and erasure

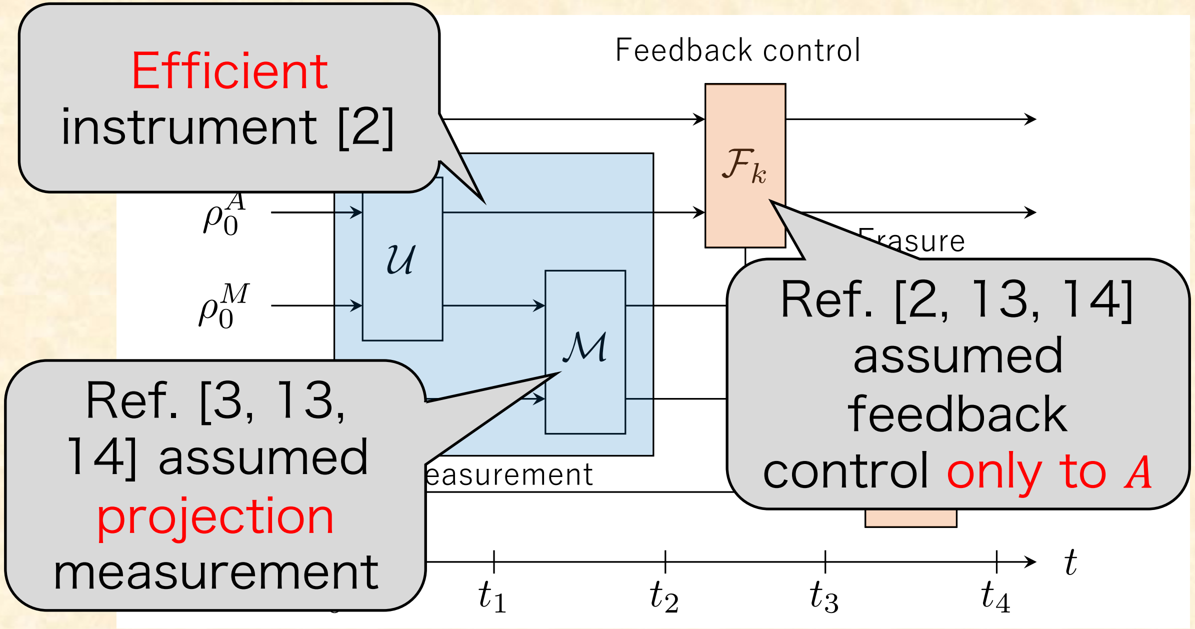

Fig. 6. Feedback control and erasure protocol

[2] T. Sagawa and M. Ueda, Phys. Rev. Lett. 100, 080403 (2008)
[3] T. Sagawa and M. Ueda, Phys. Rev. Lett. 102, 250602 (2009)
[13] K. Funo, Y. Watanabe, and M. Ueda, Phys. Rev. E 88, 052121 (2013).
[14] K. Abdelkhalek, Y. Nakata, and D. Reeb, arXiv:1609.06981 (2016).

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3. Feedback control and erasure

- Nonequilibrium 2nd law: $W_{\mathrm{ext}}^{A} - W_{\mathrm{in}}^{MK} \equiv W_{\mathrm{ext}}^{AMK} \leq -\Delta F_{0\to 4}^{AMK}$
- The 2nd law of info-thermo (Sagawa—Ueda type [3]):

$$W_{\mathrm{ext}}^{A} - W_{\mathrm{in}}^{MK} \equiv W_{\mathrm{ext}}^{AMK} \leq -\Delta F_{0\to 4}^{A}$$

[3] T. Sagawa and M. Ueda, Phys. Rev. Lett. 102, 250602 (2009).

Theorem (Universality of the 2nd law of info-thermo)
The 2nd law of info-thermo is *universal* in the sense that

Noneq. 2nd law⇛The 2nd law of info-thermo

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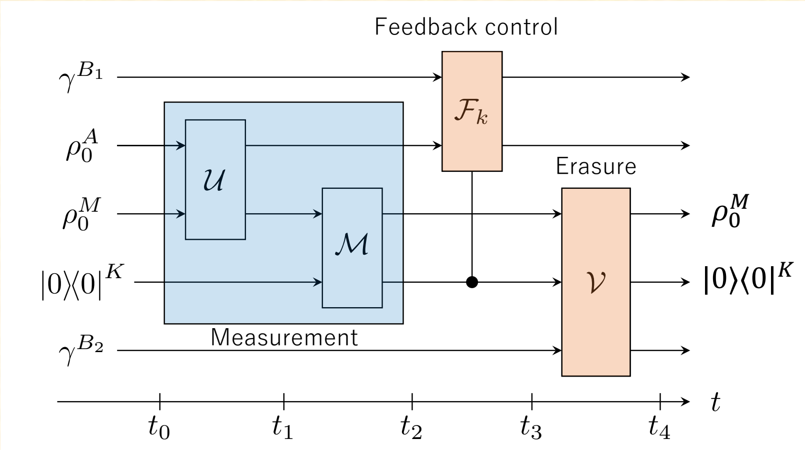
3. Feedback control and erasure



$$I_{\mathrm{GO}} := S(A)_{\rho_0} - S(A|K)_{\rho_2} \gtreqless 0 \text{ [12, 15]}$$

$$\Delta S_{0\to 2}^{AMK} := S(AMK)_{\rho_2} - S(MK)_{\rho_0} \gtreqless 0$$

$$S_{\mathrm{irr}}^{B_1} := \sum_k p_k\left(I(A{:}B_1)_{\rho_{3,k}} + D(\rho_{3,k}^{B_1}||\gamma^{B_1})\right) \geq 0$$

$$S_{\mathrm{irr}}^{B_2} := I(MK{:}B_2)_{\rho_4} + D(\rho_4^{B_2}||\gamma^{B_2}) \geq 0$$

[12] M. Ozawa, J. Math. Phys. 27, 759 (1986).
[15] H. J. Groenewold, Int. J. Theor. Phys. 4, 327 (1971).

Theorem (general work formulas)

$$W_{\mathrm{ext}}^{A} = -\Delta F_{0\to 4}^{A} + \beta^{-1}[I_{\mathrm{GO}} - (I(A{:}K)_{\rho_3} + S_{\mathrm{irr}}^{B_1})]$$

$$W_{\mathrm{in}}^{MK} = \beta^{-1}[I_{\mathrm{GO}} + \Delta S_{0\to 2}^{AMK} + I(A{:}M|K)_{\rho_2} + S_{\mathrm{irr}}^{B_2}]$$

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3. Feedback control and erasure

- Groenewold—Ozawa information gain [12, 15] vs QC-mutual information [2]

$$I_{\mathrm{GO}} \coloneqq S(A)_{\rho_0} - S(A|K)_{\rho_2} \gtrless 0$$

$$\neq I_{\mathrm{QC}} \coloneqq S\left(\rho_0^A\right) - \sum_k p_k S\left(\frac{\sqrt{\mathrm{A}_k}\rho\sqrt{\mathrm{A}_k}}{\mathrm{Tr}[\mathrm{A}_k\rho]}\right) \geq 0$$

Specialized in efficient instruments

[2] T. Sagawa and M. Ueda, Phys. Rev. Lett. 100, 080403 (2008)
[12] M. Ozawa, J. Math. Phys. 27, 759 (1986).
[15] H. J. Groenewold, Int. J. Theor. Phys. 4, 327 (1971).

3. Feedback control and erasure

Theorem 2 (general work formulas)

$$W_{\mathrm{ext}}^{A} = -\Delta F_{0\to 4}^{A} + \beta^{-1}[I_{\mathrm{GO}} - I(A{:}K)_{\rho_3} - S_{\mathrm{irr}}^{B_1}]$$
$$W_{\mathrm{in}}^{MK} = \beta^{-1}[I_{\mathrm{GO}} + \Delta S_{0\to 2}^{AMK} + I(A{:}M|K)_{\rho_2} + S_{\mathrm{irr}}^{B_2}]$$

Theorem (iff condition for the noneq. 2nd law)

$$W_{\mathrm{ext}}^{AMK} \equiv W_{\mathrm{ext}}^{A} - W_{\mathrm{in}}^{MK} \le -\Delta F_{0\to 4}^{AMK}$$
$$\Leftrightarrow \Delta S_{0\to 2}^{AMK} \ge \mathcal{J}$$
$$\mathcal{J} := I(A{:}MK)_{\rho_4} - I(A{:}M|K)_{\rho_2} - I(A{:}K)_{\rho_3} - S_{\mathrm{irr}}^{B_1} - S_{\mathrm{irr}}^{B_2} \le 0$$

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4. Derivation of the previous results

Theorem 2 (general work formulas)

$$W_{\text{ext}}^{A} = -\Delta F_{0\to4}^{A} + \beta^{-1}[I_{\text{GO}} - (I(A{:}K)_{\rho_3} + S_{\text{irr}}^{B_1})]$$
$$W_{\text{in}}^{MK} = \beta^{-1}[I_{\text{GO}} + \Delta S_{0\to2}^{AMK} + I(A{:}M|K)_{\rho_2} + S_{\text{irr}}^{B_2}]$$

Non-negative

Theorem (general work bounds)

$$W_{\text{ext}}^{A} \le -\Delta F_{0\to4}^{A} + \beta^{-1} I_{\text{GO}}$$
$$W_{\text{in}}^{MK} \ge \beta^{-1}[I_{\text{GO}} + \Delta S_{0\to2}^{AMK}]$$

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4. Derivation of the previous results

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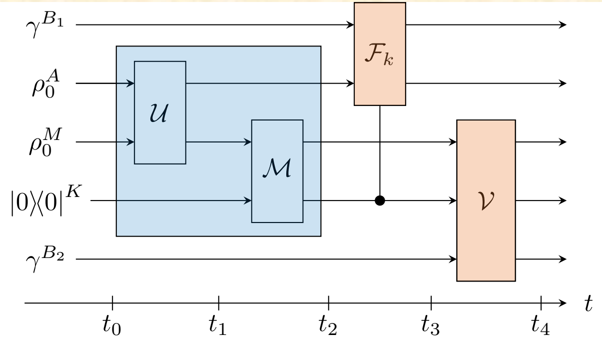
Our inequalities

$$W_{\text{ext}}^{A} \leq -\Delta F_{0\to 4}^{A} + \beta^{-1} I_{\text{GO}}$$

$$W_{\text{in}}^{MK} \geq \beta^{-1}[\Delta S_{0\to 2}^{AMK} + I_{\text{GO}}]$$

+



Sagawa and Ueda's assumption [2-4]

(A1) $\mathcal{M}$ is projective→ $\Delta S_{0\to 2}^{AMK} \geq 0$

(Use Th. 11.9 in [16])

(A2) efficient instrument →$I_{\text{GO}} = I_{\text{QC}}$ [17]

(A3) initially thermal→$-\Delta F_{0\to 4}^{A} \leq -\Delta F_{\text{eq},0\to 4}^{A}$ [8]

⇒

Sagawa and Ueda [2, 3]

$$W_{\text{ext}}^{A} \leq -\Delta F_{\text{eq},0\to 4}^{A} + \beta^{-1} I_{\text{QC}}$$

$$W_{\text{in}}^{MK} \geq I_{\text{QC}}$$

[16] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information: 10th Anniversary Edition (Cambridge University Press, 2010)

[17] F. Buscemi, M. Hayashi, and M. Horodecki, Phys. Rev. Lett. 100, 210504 (2008).

5. Conclusion

Universality of the 2nd law of info-thermo

The 2nd law of info-thermo

Noneq. 2nd law

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General work formulas)

【Iff conditions for noneq. 2nd law】

$$W_{\mathrm{ext}}^{AMK} \equiv W_{\mathrm{ext}}^{A} - W_{\mathrm{in}}^{MK} \leq -\Delta F_{0\to 4}^{AMK}$$

$$\Leftrightarrow \Delta S_{0\to 2}^{AMK} \geq \mathcal{J}$$

$$\mathcal{J} := I(A\!:MK)_{\rho_4} - I(A\!:M|K)_{\rho_2} - I(A\!:K)_{\rho_3} - S_{\mathrm{irr}}^{B_1} - S_{\mathrm{irr}}^{B_2} \leq 0$$

Acknowledgments

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